In assigning arbitrarily a function V to the interior of S, in order to get the internal density by the application of the formula (1), we may if we please discard the second of the conditions which V had to satisfy at the surface, namely, that $\frac{dV_1}{d\nu}=\frac{dV}{d\nu}$; but in that case to the mass, of finite density, determined by (1) must be added an infinitely dense and infinitely thin stratum extending over the surface, the finite superficial density of this stratum being given by (3).

We have seen that the determination of the most general internal arrangement requires the solution of the problem, To determine the potential for space external to S, supposed free from attracting matter, in terms of the given potential at the surface; and the determination of that particular arrangement in which the matter is wholly distributed over the surface, requires further the solution of the same problem for space internal to S. If, however, instead of having merely the potential given at the surface S we had given a particular arrangement of matter within S, and sought the most general rearrangement which should not alter the potential at S, there would have been no preliminary problem to solve, since V, and therefore its differential coefficients, are known for space generally, and therefore for the surface S, being expressed by triple integrals.

Instead of having the attracting matter contained within a closed surface S, and the attraction considered for space external to S, it might have been the reverse, and the same methods would still have been applicable. The problem in this form is more interesting with reference to electricity than gravitation.

II. "On the Integrability of certain Partial Differential Equations proposed by Mr. Airy." By R. Moon, M.A., late Fellow of Queen's College, Cambridge. Communicated by Professor J. J. Sylvester. Received April 30, 1867.

(Abstract.)

The equation

$$0=\frac{d^2z}{dy^2}-\alpha^2\frac{d^2z}{dx^2}+\beta\frac{dz}{dy}+\gamma z, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where α, β, γ are functions of x, includes two equations recently proposed for solution by Mr. Airy, and affords a good illustration of the ordinary incapacity of partial differential equations of the second order for solutions involving arbitrary functions.

If the above equation admit of an integral solution containing one or more arbitrary functions, it must be capable of being derived from an equation of the form

$$F(xyz)=\phi\{f(xyz)\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where F and f are definite, and ϕ is arbitrary. From this last we get

$$F'(x)+F'(z)p=\phi'(f)\{f'(x)+f'(z)p\},$$
$$F'(y)+F'(z)q=\phi'(f)\{f'(y)+f'(z)q\};$$

and eliminating ϕ' between these, we shall arrive at an equation between $xyzpq$ which is the only partial differential equation of the first order free from arbitrary functions which is obtainable from (2) without assigning to ϕ a definite form.

Hence (1) must be derivable from an equation of the form

$$f(xyzpq)=0,$$

or of the form

$$0=q+f(xyzp), \qquad (3)$$

where f is definite. Differentiating, we have

$$0=\frac{d^2z}{dy^2}+f'(y)+f'(z)q+f'(p)\frac{d^2z}{dx\,dy},$$

$$0=\frac{d^2z}{dx\,dy}+f'(x)+f'(z)p+f'(p)\frac{d^2z}{dx^2}.$$

Multiplying the second equation by A (where A is any function of $xyzpq$) and adding, we get

$$\left.\begin{aligned}0=\frac{d^2z}{dy^2}+\{f'(p)+A\}\frac{d^2z}{dx\,dy}+Af'(p)\frac{d^2z}{dx^2}+f'(y)+Af'(x)\\ +(q+Ap)f'(z).\end{aligned}\right\} \qquad (4)$$

If (1) admits of a solution of the form (2), (1) must be identical with (4), or be capable of becoming so by virtue of (3). Hence

$$0=A+f'(p), \quad -\alpha^2=Af'(p),$$
$$\beta p+\gamma z=f'(y)+Af'(x)+(Ap-f)f'(z). \qquad (5)$$

From the two first we get

$$f'(p)=\pm\alpha, \quad A=\mp\alpha.$$

Hence f must satisfy *both* the equations

$$\left.\begin{aligned}&f'(p)=\pm\alpha,\\ &0=f'(y)\mp\alpha f'(x)-(f\pm\alpha p)f'(z)-(\beta p+\gamma z).\end{aligned}\right\} \qquad (6)$$

The first gives us

$$f=F(xyz)\pm\alpha p,$$

where F is arbitrary. Substituting this value in (6), observing that

$$f'(x)=F'(x)\pm p\frac{d\alpha}{dx}, \quad f'(y)=F'(y), \quad f'(z)=F'(z),$$

we get

$$0=F'(y)\mp\alpha F'(x)-(F\pm2\alpha p)F'(z)-\left(\overline{\beta+\alpha\frac{d\alpha}{dx}}\cdot p+\gamma z\right). \qquad (7)$$

But F contains only xyz; hence, in order that this last equation may hold, the coefficient of p must $=0$, *i. e.* we must have

$$0=\pm 2\alpha F'(z)+\left(\beta+\alpha\frac{d\alpha}{dx}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

in which case (7) reduces to

$$0=F'(y)\mp\alpha F'(x)-F\,.\,F'(z)-\gamma z. \quad . \quad . \quad . \quad . \quad (9)$$

F must satisfy both (8) and (9).

By integration of (8) we get

$$F=F_1(xy)\mp\frac{1}{2}\left(\frac{\beta}{\alpha}+\frac{d\alpha}{dx}\right)z.$$

Substituting this in (9), observing that

$$F'(x)=F'_1(x)\mp\frac{1}{2}\frac{d}{dx}\left(\frac{\beta}{\alpha}+\frac{d\alpha}{dx}\right).z,$$

$$F'(z)=\mp\frac{1}{2}\left(\frac{\beta}{\alpha}+\frac{d\alpha}{dx}\right),$$

$$F'(y)=F'_1(y),$$

we get

$$\left.\begin{aligned}0=F'_1(y)\mp\alpha F'_1(x)+\frac{1}{2}\frac{d}{dx}\left(\frac{\beta}{\alpha}+\frac{d\alpha}{dx}\right)z\\ \pm\frac{1}{2}\left(\frac{\beta}{\alpha}+\frac{d\alpha}{dx}\right)\left(F_1\mp\frac{1}{2}.\overline{\frac{\beta}{\alpha}+\frac{d\alpha}{dx}}.z\right)-\gamma z\end{aligned}\right\} . \quad . \quad (10)$$

But, since F_1 contains x and y only, in order that this may hold we must have the coefficient in it of $z=0$, *i. e.*

$$0=\alpha.\frac{d}{dx}.\frac{1}{2}\left(\frac{\beta}{\alpha}+\frac{d\alpha}{dx}\right)-\overline{\frac{1}{2}\left(\frac{\beta}{\alpha}+\frac{d\alpha}{dx}\right)}^2-\gamma z,$$

in which case (10) becomes

$$0=F'_1(y)\mp\alpha F_1'(x)\pm\frac{1}{2}\left(\frac{\beta}{\alpha}+\frac{d\alpha}{dx}\right).F_1;$$

whence by integration we get

$$F_1=\epsilon^{\frac{1}{2}\int dx\left(\frac{\beta}{\alpha^2}+\frac{1}{\alpha}\frac{d\alpha}{dx}\right)}\phi\left(y\pm\int\frac{dx}{\alpha}\right),$$

where ϕ is arbitrary.

Hence we get for the first integral of (1) *when* (10) *is satisfied,*

$$0=q\pm ap\mp\frac{1}{2}\left(\frac{\beta}{\alpha}+\frac{d\alpha}{dx}\right)z+\sqrt{\alpha}\,.\,\epsilon^{\int\frac{\beta}{2\alpha^2}dx}\,.\,\left(y\pm\int\frac{dx}{\alpha}\right),$$

whence by ordinary integration we obtain the complete integral of the form,

$$z\,.\,\frac{\epsilon^{\int\frac{\beta}{2\alpha^2}dx}}{\sqrt{\alpha}}=\phi\left(y+\int\frac{dx}{\alpha}\right)+\psi\left(y-\int\frac{dx}{\alpha}\right).$$

We have above a very simple example of a general principle, viz. that in order that a partial differential equation of the second order, or a pair of simultaneous partial differential equations of the first order, may admit of a solution containing arbitrary functions, the coefficients must satisfy a certain equation of condition; from which it follows that, except in the simplest instances (in which the terms of the equation of condition vanish), there is a moral certainty that such a differential equation or pair of equations which have not been specially selected for the purpose, *and whose coefficients do not involve a disposable quantity by which the equation of condition may be satisfied,* will not admit of a solution involving arbitrary functions.

The equations applicable to the motion of an elastic fluid along the axis of a tube afford a remarkable illustration of the scope of these remarks.

Those equations consist of a pair of partial differential equations of the first order involving *five* variables, viz. y, t, ρ, v, p; and it may be shown *à priori,* that when derived upon a true theory they must be capable of a solution containing *two* arbitrary functions; from which it follows that a *third* equation will require to be satisfied. For this purpose we have p, the pressure, ready to our hands.

From the fact of the existence of the equation of condition not having been suspected by the founders of the theory of fluid-motion, at the same time that it was absolutely necessary for them to assign a form to p, they had recourse for that purpose to an empirical method; thus, on the one hand, depriving us of the power of satisfying the requirements of the problem, and on the other, abandoning the means for the determination of p which the analysis furnishes.

It cannot be matter of surprise that the law of pressure suggested under these circumstances should be entirely erroneous, as (by two other independent methods, one founded upon purely physical, the other upon purely analytical considerations) I have elsewhere shown.

III. "On the Lunar Atmospheric Tide at Melbourne." By Dr. G. NEUMAYER, late Director of the Flagstaff Observatory, Mem. Acad. Leop. Communicated by Lieut.-Gen. SABINE, President. Received April 10, 1867.

Anxious to assist the development of so interesting a branch of knowledge on the connexion of forces in nature as the influence our satellite exerts upon the earth's atmosphere, I had made it a point to include investigations, tending to facilitate studies in this direction, in the plan of discussion of the observations made at the Flagstaff Observatory about to be published. Fully aware that a geographical position, such as that of Melbourne (37° 48′ 45″ south lat. and $9^h\ 39^m\ 53^s$ east long.), affords but very few chances for arriving forthwith at a result which might be regarded as final, I thought it nevertheless of the highest importance to decide how